

A COMPARISON OF LOCALLY ADAPTIVE MULTIGRID METHODS :  
L.D.C., F.A.C. AND F.I.C.

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SUMMARY

This study is devoted to a comparative analysis of three "Adaptive ZOOM" (ZOOM Overlapping Multi-level) methods based on similar concepts of hierarchical multigrid local refinement : L.D.C. (Local Defect Correction), F.A.C. (Fast Adaptive Composite), and F.I.C. (Flux Interface Correction), which we proposed recently. These methods are tested on two examples of a bidimensional elliptic problem. We compare, for V-cycle procedures, the asymptotic evolution of the global error evaluated by discrete norms, the corresponding local errors, and the convergence rates of these algorithms.

INTRODUCTION

The need for local resolution in physical models occurs frequently in practice. Special local features of the operator coefficients, source terms, and boundary conditions can demand resolution in restricted regions of the domain that is much finer than the required global resolution. The multigrid methods with local mesh refinement provide one solution method to achieve efficient local resolution by solving problems on various locally nested

grids, and by using these grids as a basis for fast solution and correction on the global basic grid of the calculation domain. Different techniques have been proposed in the literature, such as the pioneering works [1,2,3,4,5].

Therefore, the concept of "Computational Adaptive Zoom" in the context of a "Graphical and Computational Architecture" has been introduced in the field of numerical simulation in order to take the best advantage of the new capabilities of high performance computer architectures [6]. It can be viewed as a generation made automatically (i.e. in an adaptive way) or not, of some multilevel hierarchical local nested zoom grids (ZG), overlapped all over the global basic grid (BG). These grids may move all over the entire computation domain  $\Omega$  during the solution phase. This concept is supposed to allow both local refinement and global correction of the basic grid solution by a successive transfer of information between the connected grids (BG) and (ZG). So it is well adapted to a graphical vision of Zoom in terms of the creation of local graphical windows where it is needed in the problem (strong gradients, discontinuities, singularities,...), but in an active sense, i.e., the basic grid solution is modified and improved as the computing is performed. This has involved us in the creation of an original engineering software package called "AQUILON", still currently in development [6].

In addition, this strategy offers other interests. The goal is to combine the best features of both multigrid techniques and domain decomposition methods (in the case of overlapping grids) to provide an acceleration of the convergence rate and a good suitability for implementation on parallel computers, thus reducing the ellapse time. Moreover, another advantage is the possibilty to solve different differential problems on the grids (BG) and (ZG), which allows us to optimize both the physical and the numerical model. This can be particularly interesting for the approach of solving problems by "imbedding inside fictitious domains" associated with appropriate "control terms" for expressing the boundary conditions, as proposed in [6]. It is also possible to adopt different kinds of discretization on each grid. Thereby, the multigrid zoom methods share with the domain decomposition techniques the opportunity for obtaining precise solutions by combining solutions to problems posed on physical subdomains, or, more generally, by combining solutions to appropriately constructed continuous and discrete boundary value sub-problems.

From the numerical point of view, the strategy adopted enables us to work only on structured and uniform meshes for each grid separately, on which a moderate number of degrees of freedom is required. On each grid, a "simple and

inexpensive" discretization is performed, leading to the same simple form of sparse pattern matrices (e.g. 2D block-tridiagonal). We aim at avoiding solving problems on unstructured or nonuniform composite meshes, which tends to introduce inaccuracies in the discretization, slowness in the solvers, and being surely more expensive in terms of implementation, data structures storage and CPU time. Our choice is expected to be relatively good in terms of duality quality/cost of computation for a lot of cases of moderate complexity.

## MULTIGRID ZOOM ALGORITHMS

Different ZOOM algorithms will be examined and compared. We consider first the L.D.C. (Local Defect Correction) algorithm proposed by Hackbush [1]; we choose for the restriction operator a 2D bilinear interpolation one of type "full weighting control volume". The second one belongs to the class of F.A.C. (Fast Adaptive Composite Grid) methods from McCormick [5], for which the analogy with the B.E.P.S. method [4] can be noticed. We use here the "delayed correction" version of F.A.C. Only the third one, the F.I.C. (Flux Interface Correction) algorithm that we proposed more recently [7], will be briefly described hereafter.

All these Multigrid Zoom Algorithms are based on the same general principle : a successive transfer of information level by level, leading to the global correction of the initial discrete solution on each grid, and thus on the global basic grid (BG). The multilevel implementation is made in a recursive way as in the usual multigrid techniques (V-Cycles, W-Cycles, etc .) [1,3]. The resolution on each grid may be performed "exactly" or by using an inexact solve (e.g. a few iterations of a smoothing procedure).

## Notations and Definitions

Consider the following second order non-linear elliptic boundary value problem defined on  $\Omega$  a bounded, open domain in  $\mathbb{R}^d$ , for  $d = 2$  or  $3$  :

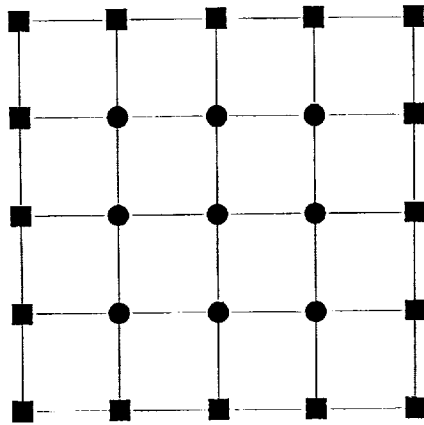
$$(\mathcal{P}) \begin{cases} L(u) \equiv \text{div}(\varphi(u)) + G(u) = f(x) & x \in \Omega \\ \text{well-posed boundary conditions on } \Gamma = \partial\Omega \end{cases} \quad \text{symbolically called by (BC)} \quad (1)$$

The equation (1)  $L(u) \equiv f$  is so expressed by splitting the nonlinear operator  $L(u)$  in the divergent part where  $\varphi(u)$  has the physical meaning of the flux density of the solution  $u = u(x)$  and the nonconservative one  $G = G(u)$ . The relation between the solution  $u$  and the flux  $\varphi$  can take the general vector

form  $\varphi(u) = F(u)$  in many systems of conservation laws, but applications will concern an advection-diffusion equation or a Navier-Stokes problem. For the experiments here,  $(\mathcal{P})$  is a diffusion problem and we have  $\varphi(u) = -\sigma \cdot \text{grad} u$ .

In order not to have too many formal requirements and restrictions, we assume explicitly only that this equation (1) has at least one isolated solution  $u^*$  in the space  $L^2(\Omega)$ . All other assumptions are implicitly contained in the following considerations.

The basic notations will be those classically used in the multigrid framework [1]. We denote by  $\ell$  the current index of the grid level ( $0 \leq \ell \leq \ell^*$ ),  $\ell = 0$  is the level of the global basic grid (BG) which discretizes the entire calculation domain  $\Omega$ , and  $\ell = \ell^* \neq 0$  is the level of the most nested and finest zoom grid (ZG). Each grid of level  $\ell$  can be characterized by :



- the open domain  $\Omega_\ell = \{ \bullet \}$
- the boundary  $\Gamma_\ell = \{ \blacksquare \}$  on which can be defined the unit outside normal vector  $\mathbf{n}_\ell$
- the closure  $\overline{\Omega}_\ell = \Omega_\ell \cup \Gamma_\ell$
- the mesh size  $h_\ell$

Each grid of level  $\ell$  is divided into a set of control volumes  $V_x$  associated to the nodes  $x \in \overline{\Omega}_\ell$ . We denote by  $\Gamma_{\ell, \ell+1}$  the interface between two successive grids of level  $\ell$  and  $\ell+1$  and we have  $\forall \ell, \Omega_\ell \cap \Omega_{\ell+1} \neq \emptyset$ . The successive mesh sizes will be taken as  $h_{\ell+1} = h_\ell / 2^p$ ,  $p \in \mathbb{N}^*$ . The following notations will also be used :  $A_\ell = \Omega_{\ell+1} \cap \Omega_\ell$  and  $\overline{A}_\ell = \overline{\Omega}_{\ell+1} \cap \Omega_\ell$ .

The transfer operators between the grids  $\ell$  and  $\ell+1$  will be called, respectively, by  $R_{\ell+1}^\ell$  for the restriction operator and by  $P_\ell^{\ell+1}$  for the prolongation operator. For all three algorithms, we have chosen  $P_\ell^{\ell+1}$  as :

$$P_\ell^{\ell+1} : \Gamma_{\ell, \ell+1} \cap \overline{\Omega}_\ell \longrightarrow \Gamma_{\ell+1} \setminus (\Gamma_{\ell+1} \cap \Gamma_\ell)$$

which is a monodimensional linear interpolation operator defined on the interface of the grids  $\ell$  and  $\ell+1$ . Each value  $u_{\ell+1}$  at a node  $y \in \Gamma_{\ell+1} \setminus (\Gamma_{\ell+1} \cap \Gamma_\ell)$  on the interface is obtained by a linear interpolation of the values  $u_\ell$  at the two neighbour nodes  $x$  et  $x'$  belonging to  $(\Gamma_{\ell, \ell+1} \cap \overline{\Omega}_\ell)$ , and thus verifying  $u_{\ell+1}(y) = u_\ell(x)$  if  $y = x$ .

If we denote by  $L_\ell u_\ell = f_\ell$  the discretized equation of (1) on the grid of level  $\ell$ , we can define the following discrete boundary value problems on  $\bar{\Omega}_\ell$ :

$$(\mathcal{P}_0) \left\{ \begin{array}{l} L_0 u_0 = f_0 : \text{in } \Omega_0 \\ \text{on } \Gamma_0 : (\text{BC}) \end{array} \right\} \quad (\mathcal{P}_\ell) \left\{ \begin{array}{l} L_\ell u_\ell = f_\ell : \text{in } \Omega_\ell \\ \text{on } \Gamma_\ell \cap \Gamma : (\text{BC}) \\ \text{on } \Gamma_\ell \setminus (\Gamma_\ell \cap \Gamma) : u_\ell = P_{\ell-1}^\ell u_{\ell-1} \end{array} \right\} \quad (2) \quad \ell \neq 0$$

We denote by  $u_\ell^k$  the discrete solution obtained on the grid  $\bar{\Omega}_\ell$  at the  $k$ -th iteration of the zoom algorithm, and  $e_\ell^k = u_\ell^* - u_\ell^k$  the associated discrete error, where  $u_\ell^*$  is the natural restriction of the exact solution  $u^*$  of problem  $(\mathcal{P})$  on  $\bar{\Omega}_\ell$ .

For  $0 < \ell < \ell^*$ ,  $\gamma(\ell)$  will represent the number of iterations of the zoom algorithm on the grid level  $\ell$  in order to describe a whole cycle: if we have  $\gamma(\ell)=1$  (respectively  $\gamma(\ell)=2$ ),  $\forall 0 < \ell < \ell^*$ , then V-cycles (respectively W-cycles) will be described. We have  $\gamma(\ell^*) = 1$ , and  $\gamma(0)$  is the total number of cycles performed from the basic grid (BG) in order to obtain the so-called convergence of the zoom algorithm. When  $\ell^*=1$  (i.e. for a two-grid algorithm), only V-cycles are of course carried out. The term "No Zoom" will be used for the resolution by "an exact solve" of problem  $(\mathcal{P})$  on the basic grid (BG) of mesh size  $h_0$  ( $\ell^*=0$ ,  $k=0$ ). The term "Zoom" will be used to indicate that some iterations of a multilevel zoom algorithm have been performed:  $\ell^* \neq 0$ ,  $1 \leq k \leq \gamma(0)$ .

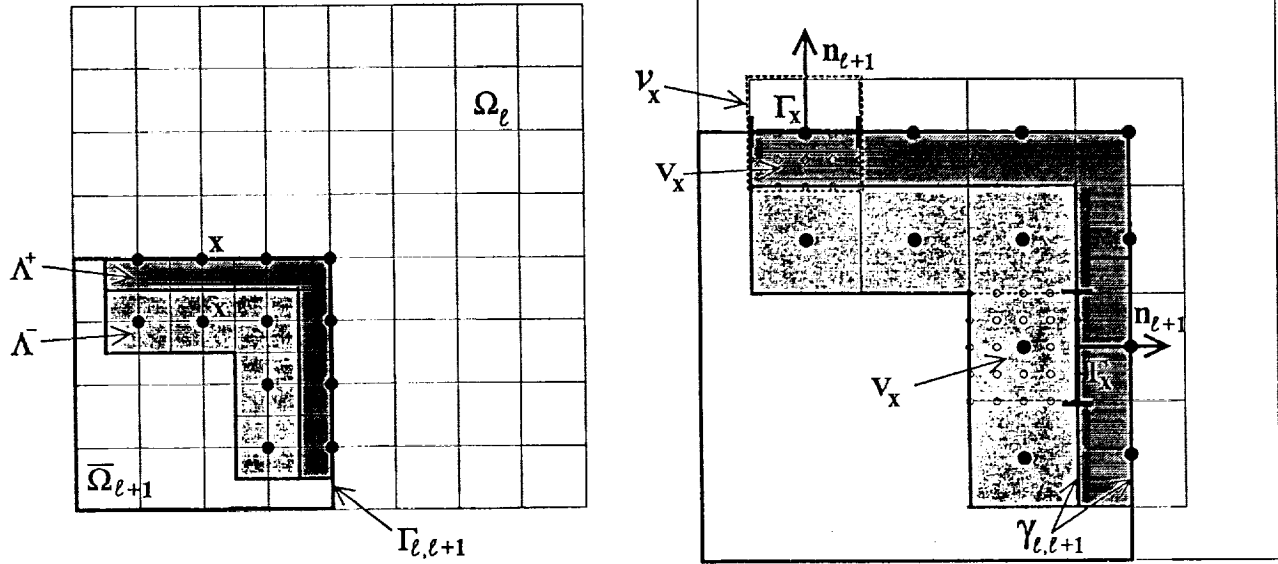
### Description of The Multilevel F.I.C. Algorithm

The main idea of the two-grid FIC method for levels  $\ell$  and  $\ell+1$  is to give the opportunity to apply the local "flux residual" correction due to the whole patch level  $\ell+1$ , at each node  $x \in \bar{A}_\ell$  on the grid level  $\ell$ . This is obtained through the expression of the local flux balance (i.e. integration) of eq.(1) over the volume  $V_x = v_x \cap \bar{\Omega}_{\ell+1}$ , between the grid levels  $\ell$ , on one hand, and  $\ell+1$  on the other hand.

Because of the consistency of the conservative discretization of the fluxes by the finite volume method, which must be respected on each grid, the outside normal fluxes of  $\phi(u)$  through an interface of two neighbour control volumes are opposite. By giving more importance to the local "flux residual", that leads to consider for the correction step on the grid level  $\ell$ , the local flux of the defect only at each node of a boundary zone  $I_{\ell,\ell+1}$  defined as the "flux correction interface". We can choose for  $I_{\ell,\ell+1}$ , either the stripe  $\Lambda^+ = \{\cup V_x, x \in \partial \bar{A}_\ell = \Gamma_{\ell,\ell+1} \cap \Omega_\ell\}$ , or  $\Lambda^-$  (see further Figure) if we want the

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boundary  $\partial\Lambda$  to correspond to interfaces between control volumes on the grid level  $\ell$  : we will have  $V_x = v_x$  in the latter case. We define on the grid level  $\ell$ ,  $\partial V_x = \Gamma_x \cup \Gamma_r$ ,  $\forall x \in I_\ell = I_{\ell,\ell+1} \cap \Omega_\ell$ , where  $\Gamma_x = \partial V_x \cap \partial\bar{\Lambda}_\ell \neq \emptyset$ , or respectively,  $\Gamma_x = \partial V_x \cap \partial A_\ell$ , ( $\text{mes}(\Gamma_x) = h_\ell$ ).



We then propose the following restriction operator on the outside normal flux through the "interface boundary"  $\gamma_{\ell,\ell+1} = \{\cup \Gamma_x, x \in I_\ell\}$  :

$$R_{\ell+1}^\ell : \gamma_{\ell,\ell+1} \cap \bar{\Omega}_{\ell+1} \longrightarrow I_{\ell,\ell+1} \cap \Omega_\ell$$

$$R_{\ell+1}^\ell(\phi_{\ell+1}(u) \cdot n_{\ell+1})(x) = \frac{1}{\text{mes}(\Gamma_x)} \int_{\Gamma_x} \phi_{\ell+1}(u) \cdot n_{\ell+1} d\gamma \quad \forall x \in I_{\ell,\ell+1} \cap \Omega_\ell \quad (3)$$

We can then define, as in [7], the local "flux residual" correction at each node  $x \in I_\ell = I_{\ell,\ell+1} \cap \Omega_\ell$  on the grid level  $\ell$  by :

$$r_\ell(\phi)(x) = \frac{\omega(\ell, x)(u)}{\varepsilon(\ell, x)} \left\{ R_{\ell+1}^\ell(\phi_{\ell+1}(u) \cdot n_{\ell+1}) - \phi_\ell(u) \cdot n_{\ell+1} \right\} (x) \quad (4)$$

The control parameter  $\varepsilon(\ell, x)$ , which has the dimension of a length, has already been encountered in order to assign Neumann and Robin (or Fourier) boundary conditions in the context of "imbedding inside a fictitious domain" [6]. Its expression is given by :

$$\varepsilon(\ell, x) = \frac{\text{mes}(V_x)}{\text{mes}(\Gamma_x)} \quad (5)$$

A complete calculation, still not published, gives a complex expression for  $\omega(\ell, x)(u)$ , which is the following one in the case under subject of  $G \equiv 0$  :

$$\omega(\ell, \mathbf{x})(u)^* \approx 1 + \frac{\left( \int_{\Gamma_r} \varphi(u) \cdot \mathbf{n} \, d\gamma \right)_{\ell+1} - \left( \int_{\Gamma_r} \varphi(u) \cdot \mathbf{n} \, d\gamma \right)_{\ell}}{\left( \int_{\Gamma_x} \varphi(u) \cdot \mathbf{n} \, d\gamma \right)_{\ell+1} - \left( \int_{\Gamma_x} \varphi(u) \cdot \mathbf{n} \, d\gamma \right)_{\ell}} \quad (6)$$

We can then generate the successive iterates  $u_0^k$  by the multilevel FIC algorithm implemented in a recursive way :

**Initialization** : compute  $u_0^0$   
 $u_0^0$  is obtained by resolution of problem  $(\mathcal{P}_0)$   
**Iterations** : compute the successive iterates  $u_0^k$   
for  $k = 1$  to  $\gamma(0)$  do FIC(0)  
**Composite re-actualization** : providing  $u_0^{\gamma(0)}$  on  $(BG)$  by assigning  
for  $\ell = \ell^* - 1$  to 0 by step of -1 :  $u_{\ell}^{\gamma(0)}(\mathbf{x}) = u_{\ell+1}^{\gamma(0)}(\mathbf{x}) \quad \forall \mathbf{x} \in A_{\ell}$   
**Procedure FIC( $\ell$ )**

If  $\ell = \ell^*$  Then solve problem  $(\mathcal{P}_{\ell^*})$  Else  
begin  
\* 1<sup>st</sup> step - resolution on the grid level  $\ell+1$  :  
- solve problem  $(\mathcal{P}_{\ell+1})$  providing  $u_{\ell+1}$   
- for  $k = 1$  to  $\gamma(\ell+1)$  do FIC( $\ell+1$ )  
\* 2<sup>nd</sup> step - correction on the grid level  $\ell$  :  
- solve problem  $(\mathcal{P}_{\ell})$  with  $\bar{f}_{\ell} = f_{\ell} + \chi_{I_{\ell}} r_{\ell}(\varphi)$   
where  $r_{\ell}(\varphi)$  is computed by equations (3)(4)(5)(6)  
and  $\chi_{I_{\ell}}$  is the characteristic function of  $I_{\ell}$  in  $\Omega_{\ell}$   
end

*Remarks :*

1) - In any case, in order to avoid the explicit calculation of  $\omega(\ell, \mathbf{x})(u)$  by eq.(6), an economical solution is to use an approximate correction for FIC. In that version, called FIC( $\omega$ ), only the flux integrals on the interface  $\Gamma_x$  will be evaluated by quadrature formulae (Simpson), and an average weighting factor  $\omega(\ell)$  will be determined by a semi-empirical way for each grid level. Besides, it can play the role of an average relaxation parameter for the iterative zoom algorithm when  $\omega(\ell) = \omega, \forall \ell \neq \ell^*$ .

2) - In terms of domain decomposition, the two-grid FIC for levels  $\ell$  and  $\ell+1$  can be regarded as a full overlapping iterative algorithm that splits the whole composite problem in two Dirichlet/ Neumann boundary value sub-problems:

- the problem on the grid level  $\ell+1$  with a Dirichlet boundary condition on the interface  $\Gamma_{\ell, \ell+1}$  (2),
- the problem on the level  $\ell$  with a condition of relaxed transmission of the flux on the interface  $\gamma_{\ell, \ell+1}$  through (4), which demands the flux continuity at convergence. That condition can be considered as a Neumann boundary condition on  $\gamma_{\ell, \ell+1}$  by the technique of "fictitious domain" in [6].

### General Comments on the Three Algorithms

i) - The two-grid FAC method for levels  $\ell$  and  $\ell+1$  can be regarded as an iterative procedure to solve "exactly" the discrete composite problem coming from an adequate discretization of problem  $(\mathcal{P})$  on the composite grid  $\bar{\Omega}_{\ell}$  defined by the association of the grids  $\bar{\Omega}_{\ell}$  and  $\bar{\Omega}_{\ell+1}$ . Therefore, the principle is to apply a multigrid algorithm between the grids  $\bar{\Omega}_{\ell}$  and  $\bar{\Omega}_{\ell}$  on one hand, and between the grids  $\bar{\Omega}_{\ell}$  and  $\bar{\Omega}_{\ell+1}$  on the other [5,4]. There is therefore a correction phase on both the grid levels  $\ell$  and  $\ell+1$  with respect to the discretization on the composite grid. In that sense, FAC can be viewed as an "exact" solver for the composite problem. Because the composite grid stencils agree with the coarse and fine grid stencils, respectively, outside and inside the refinement region, and because the correction equations are solved exactly, the composite grid residual is nonzero only at the interface.

ii) - Due to the attention needed for the nonuniform discretization of the problem on the interface zone of the composite grid, FAC method can prove to be a little difficult to implement in a more than two grids version.

iii) - On the contrary, LDC and FIC methods, which are easier implementing in the multilevel case, are only approximate solvers : they don't use a composite grid and neglect the fine grid residual correction. The former consists in the local correction of the solution defect inside  $A_{\ell}$  as the latter involves a local flux residual correction through the interface  $\gamma_{\ell, \ell+1}$ .

iv) - Both FAC and FIC methods provide corrections by balancing fluxes computed from both coarse and fine grids across the interface. They take the best advantage of a conservative discretization of the equations, for example, by a finite volume technique.



## NUMERICAL APPLICATIONS

In that context, we propose to compare three types of multigrid zoom algorithms on two examples of a linear elliptic problem  $(\mathcal{P})$  presenting, respectively, a discontinuity of the operator coefficients for  $(\mathcal{P}1)$  [8], and a singularity of the exact solution for  $(\mathcal{P}2)$  [1] :

$$(\mathcal{P}) \begin{cases} L(u) \equiv -\text{div}(\sigma(x) \cdot \text{grad} u) + \alpha(x) u = f(x) & \text{in } \Omega = ]0,1[ \times ]0,1[ \quad (1') \\ \sigma, \alpha > 0 \in L^\infty(\Omega) \text{ et } f \in L^2(\Omega) \\ \text{well-posed boundary conditions on } \Gamma = \partial\Omega \text{ symbolically called by (BC)} \end{cases}$$

These problems were already tested successfully on the FIC method in [7]. Problem  $(\mathcal{P}1)$  is heterogeneous and defined by  $f \equiv 0$ ,  $\alpha \equiv 0$ ,  $\sigma \equiv 100$  inside a disk of radius 0.1 and  $\sigma \equiv 1$  outside (Fig.1a). A solution computed on a very fine basic mesh ( $512^2$ ) will be used as the reference solution  $u^*$ . Problem  $(\mathcal{P}2)$  is defined by  $f \equiv 0$ ,  $\alpha \equiv 0$ ,  $\sigma \equiv 1$  (Fig.1b); the exact solution is  $u^* = \ln(r)$  with  $r = \sqrt{x^2 + y^2}$ .

### Numerical Implementation and Procedures

The discretization on each grid, independant of the geometry of the problem, is made in a conservative way by a finite volume method on a uniform Cartesian mesh. The classical five-point scheme is used providing a second order precision. The resolution of the linear systems, which are block-tridiagonal and symmetric positive definite, is performed by a fast and efficient solver : a preconditionned conjugate gradient (PCG) method (CG-SSOR) vectorized by a Red and Black numbering of unknowns. The results for two grids are obtained by an "exact" solve on each grid. The results for multilevel LDC or FIC ( $\ell^* \geq 1$ ) are given for an "inexact" solve on each grid (including Fig.5b), i.e., a fixed number  $itcg$  of iterations of PCG on each grid with :

$$itcg=2 \text{ for } h_0 = 1/8 \qquad itcg=4 \text{ for } h_0 = 1/16 \qquad itcg=8 \text{ for } h_0 = 1/32$$

The results are analyzed with different norms ( $L^\infty$ ,  $L^2$ , L-energy norm) of the discrete error evaluated on the global basic grid (BG,  $\ell=0$ ). We study the asymptotic evolution of the relative error norms  $\xi_0^0 = \|e_0^0\| / \|u_0^*\|$  (No Zoom) and  $\xi_0^{\gamma(0)} = \|e_0^{\gamma(0)}\| / \|u_0^*\|$  (after  $\gamma(0)$  Zoom iterations) with  $e_0^k = u_0^k - u_0^*$ , which allows us to estimate an asymptotic average rate  $\tau$  :

\* for  $\ell^* = 1$ , as function of  $h_1$  or  $p$  (for a fixed  $h_0$ )

$$\tau = \left( \frac{\xi_0^0}{\xi_0^{\gamma(0)}(p=m)} \right)^{1/m}, \text{ with } m = \max \left\{ p \in \mathbb{N}^* \right\}$$

Here  $m=3$  and  $\gamma(0)=2$ , see Tab.1, Tab.2, and Fig.2a, Fig.3, and Fig.4.

\* for  $\ell^* \geq 1$ , as function of  $\ell^*$  (for a fixed  $h_0$  and  $p=1$ )

$$\tau = \left( \frac{\xi_0^0}{\xi_0^{\gamma(0)}(\ell^*=m)} \right)^{1/m}, \text{ with } m = \max \left\{ \ell^* \in \mathbb{N}^* \right\}$$

Here  $m=3$  and  $\gamma(0)=10$ , see Tab.3, and Fig.2b.

The convergence rate of LDC, FAC and FIC have been also compared (Tab.4):

\* for FAC : we study the variations of the Euclidean norm of the composite residual  $|r_0^k(u)|_2$  for  $k = 1$  to  $\gamma(0)$  (Fig.5a), and a convergence rate  $\rho$  is calculated by a geometric mean :

$$\rho = \left( \frac{|r_0^{\gamma(0)}(u)|_2}{|r_0^1(u)|_2} \right)^{1/(\gamma(0)-1)}$$

\* for LDC or FIC: we study the variations of quantities  $\delta_0^k = \|u_0^k - u_0^{k-1}\|_{L^2}$  for  $k = 1$  to  $\gamma(0)$  (Fig.5b), and a convergence rate  $\rho$  is then estimated by :

$$\rho = \left( \frac{\delta_0^{\gamma(0)}}{\delta_0^1} \right)^{1/(\gamma(0)-1)}$$

### Comparative Numerical Results

1) - By comparing a no-zoom method and a ZOOM one, we notice that the error globally decreases ; between two increments of  $p$  or  $\ell^*$ , it is divided by an elevated average  $\tau$ -factor of between 1.5 and 3.5 (Tab.1, Tab.2, Tab.3). For problem (P2), the decrease is monotonic and there seems to be good analogy between the variation of the error as a function of  $p$  (for  $\ell^*=1$ ) or of  $\ell^*$  (for  $p=1$ ), (see Fig.2a and Fig.2b). The influence of the position and dimensions of the local grids (ZG) becomes negligible as  $h_0$  decreases [7]. Due to the choice

of discretizing on a Cartesian mesh independently of the geometry of the heterogeneity, the error for problem (P1) does not decrease monotonically as already noticed in [7,8].

2) - In many cases, the error obtained with zoom is less than computed without zoom on a single basic grid of mesh size  $h_0 \leq h_\ell^*$ . In particular, Fig.4 shows that the local discrete error  $|e_\ell^k|$  calculated point by point on the diagonal of the domain (P2), by a two-grid FIC method ( $h_0=1/16$ ,  $h_1=h_0/2$ ,  $k=2$ ) is globally better than the error obtained with No Zoom at the corresponding nodes of BG ( $\ell=0$ ,  $h_0=1/32$ ). The former results are more accurate inside the refinement region and get closer to the latter case far from the interface. Such remarks can also be made for the discrete error norms in the other Tables or Figures. However, the error is not reduced beyond a threshold value consistent with the order of precision of the discretization schemes on the different grids (cf, the multigrid defect correction method using Richardson extrapolation [1]).

3) - The two-grid FAC and FIC methods yield error results of the same order of magnitude for both problems. These results are far better than for LDC for problem (P1), where the flux conservation plays an important role. On the contrary, LDC yields as good results as the others for problem (P2), and sometimes better. However, as LDC does not deal with the interface fluxes, but only works on the solution inside the open refinement region, it can become inefficient ( $\tau = 1$ ) if the refinement region does not contain enough coarse nodes on which the local defect correction is performed (Tab.1, Tab.2, Tab.3).

4) - The results with the version  $FIC(\omega)$  for  $0,1 \leq \omega \leq 0,5$  are nearly similar to those obtained with  $\omega^* = \omega(\ell, x)(u)$  calculated by (6) (Fig.3). That could justify the interest of the approximate version  $FIC(\omega)$ , and particularly as a preconditioner of the discrete problem, as suggested in [4].

5) - Because of its exact character, the FAC method yields the far best convergence rate, a mean value of 0.16, nearly independant of both  $h_0$  and  $h_1$  (Fig.5a and Tab.4). We obtain a mean convergence rate of 0.42 for  $FIC(\omega=0.2)$ , just a little better than LDC with 0.50. These convergence rates remain not very sensitive to the variations of  $h_0$  and  $\ell^*$  (Fig.5b and Tab.4). However, those of FIC have a noticeable tendency to become better as the number of grid levels (or  $\ell^*$ ) increases (see Tab.4).

## CONCLUSION

Despite its non-exact character, FIC provides as good results as FAC, concerning the analysis of discrete errors for both the two tested problems. In particular, FAC and FIC proved to be better than LDC for problems where the flux conservation locally plays a main role.

FAC yields very good convergence rates ( $\rho \approx 0.16$ ), better than LDC ( $\rho \approx 0.50$ ) or FIC ( $\rho \approx 0.42$ ), but its multilevel implementation remains more difficult. However, the use of FIC as a preconditioning technique of the discrete problem is likely to be very interesting, especially for the approximate version  $FIC(\omega)$  where the factor  $\omega$  becomes a relaxation parameter. We are currently testing such a procedure for Navier-Stokes problems.

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Tab. 1. Problem (P1) - Two Grid Zoom  $\gamma(0) = 2$  - Discrete  $L^2$  norm of the error

$h_0$	NO ZOOM	$h_1 = \frac{h_0}{2^p}$	ZOOM $x_1=0.375$ et $x_2=0.625$			ZOOM $x_1=0.25$ et $x_2=0.75$		
			LDC	FAC	FIC $\omega = 0.55$	LDC	FAC	FIC $\omega = 0.40$
1/8	0.434E-1	p=1	0.434E-1	0.493E-1	0.210E-1	0.434E-1	0.732E-2	0.639E-2
		p=2	0.434E-1	0.790E-1	0.122E-1	0.434E-1	0.133E-1	0.106E-1
		p=3	0.434E-1	0.330E-1	0.589E-2	0.434E-1	0.289E-2	0.251E-2
		$\tau$	1.00	1.10	1.95	1.00	2.47	2.59
1/16	0.689E-2	p=1	0.689E-2	0.167E-1	0.119E-1	0.739E-2	0.986E-2	0.106E-1
		p=2	0.689E-2	0.374E-2	0.394E-2	0.297E-2	0.123E-2	0.161E-2
		p=3	0.689E-2	0.255E-2	0.339E-2	0.214E-2	0.673E-3	0.592E-3
		$\tau$	1.00	1.39	1.27	1.48	2.17	2.27
1/32	0.982E-2	p=1	0.136E-1	0.342E-2	0.538E-2	0.244E-2	0.160E-2	0.201E-2
		p=2	0.140E-1	0.295E-2	0.412E-2	0.174E-2	0.428E-3	0.392E-3
		p=3	0.134E-1	0.326E-2	0.331E-2	0.153E-2	0.233E-3	0.262E-3
		$\tau$	0.90	1.44	1.44	1.86	3.48	3.35
1/64	0.192E-2							
1/128	0.648E-3							
1/256	0.193E-3							

Tab. 2. Problem (P2) - Two Grid Zoom  $\gamma(0) = 2$  - Discrete L-Energy norm of the error

$h_0$	NO ZOOM	$h_1 = \frac{h_0}{2^p}$	ZOOM $x_1=0$ et $x_2=0.25$			ZOOM $x_1=0$ et $x_2=0.5$		
			LDC	FAC	FIC $\omega = 0.20$	LDC	FAC	FIC $\omega = 0.20$
1/8	0.342E-1	p=1	0.342E-1	0.152E-1	0.168E-1	0.120E-1	0.138E-1	0.139E-1
		p=2	0.342E-1	0.817E-2	0.110E-1	0.441E-2	0.511E-2	0.553E-2
		p=3	0.342E-1	0.622E-2	0.990E-2	0.221E-2	0.250E-2	0.346E-2
		$\tau$	1.00	1.77	1.51	2.49	2.39	2.15
1/16	0.206E-1	p=1	0.723E-2	0.829E-2	0.837E-2	0.704E-2	0.818E-2	0.819E-2
		p=2	0.270E-2	0.308E-2	0.337E-2	0.232E-2	0.282E-2	0.285E-2
		p=3	0.140E-2	0.152E-2	0.215E-2	0.627E-3	0.975E-3	0.107E-2
		$\tau$	2.45	2.38	2.12	3.20	2.76	2.68
1/32	0.133E-1	p=1	0.454E-2	0.527E-2	0.527E-2	0.453E-2	0.526E-2	0.526E-2
		p=2	0.150E-2	0.182E-2	0.184E-2	0.149E-2	0.180E-2	0.180E-2
		p=3	0.407E-3	0.628E-3	0.699E-3	0.387E-3	0.570E-3	0.577E-3
		$\tau$	3.20	2.77	2.67	3.25	2.86	2.85
1/64	0.888E-2							
1/128	0.607E-2							
1/256	0.420E-2							
1/512	0.294E-2							

Tab. 3. Problem (P2) - Multilevel Zoom LDC/FIC

$\gamma(0) = 10$ ,  $h_{\ell+1} = h_{\ell}/2$   $0 \leq \ell \leq \ell^* - 1$ ,  $x_1 = 0$  and  $x_2 = 0.5$  -  
Discrete  $L^2$  norm of the error

$h_0$	NO ZOOM $\ell^* = 0$	number of grids	ZOOM LDC	ZOOM FIC	
				$\omega = 0.2$	$\omega = 0.35$
1/8	0.209E-1	$\ell^* = 1$	0.806E-2	0.869E-2	0.787E-2
		$\ell^* = 2$	0.440E-2	0.625E-2	0.465E-2
		$\ell^* = 3$	0.368E-2	0.606E-2	0.428E-2
		$\tau$	1.78	1.51	1.70
1/16	0.105E-1	$\ell^* = 1$	0.399E-2	0.404E-2	0.394E-2
		$\ell^* = 2$	0.157E-2	0.199E-2	0.166E-2
		$\ell^* = 3$	0.102E-2	0.165E-2	0.115E-2
		$\tau$	2.18	1.85	2.09
1/32	0.529E-2	$\ell^* = 1$	0.186E-2	0.202E-2	0.249E-2
		$\ell^* = 2$	0.969E-3	0.768E-3	0.809E-3
		$\ell^* = 3$	0.711E-3	0.416E-3	0.514E-3
		$\tau$	1.95	2.33	2.18
1/64	0.265E-2				
1/128	0.132E-2				
1/256	0.662E-3				
1/512	0.331E-3				

Tab. 4. Problem (P2) - Multilevel Zoom LDC/FIC - Two Grid Zoom FAC

$\gamma(0) = 10$ ,  $h_{\ell+1} = h_{\ell}/2^p$   $0 \leq \ell \leq \ell^* - 1$ ,  $x_1 = 0$  and  $x_2 = 0.5$  -  
Convergence rate  $\rho$

$h_0$	number of grids	ZOOM LDC	ZOOM FIC $\omega = 0.2$	ZOOM FAC		
		$p = 1$	$p = 1$	$p=1$	$p=2$	$p=3$
1/8	$\ell^* = 1$	0.53	0.46	0.14	0.15	0.15
	$\ell^* = 2$	0.45	0.42			
	$\ell^* = 3$	0.43	0.34			
1/16	$\ell^* = 1$	0.56	0.44	0.14	0.16	0.16
	$\ell^* = 2$	0.49	0.42			
	$\ell^* = 3$	0.48	0.39			
1/32	$\ell^* = 1$	0.50	0.47	0.15	0.17	0.18
	$\ell^* = 2$	0.51	0.44			
	$\ell^* = 3$	0.50	0.40			

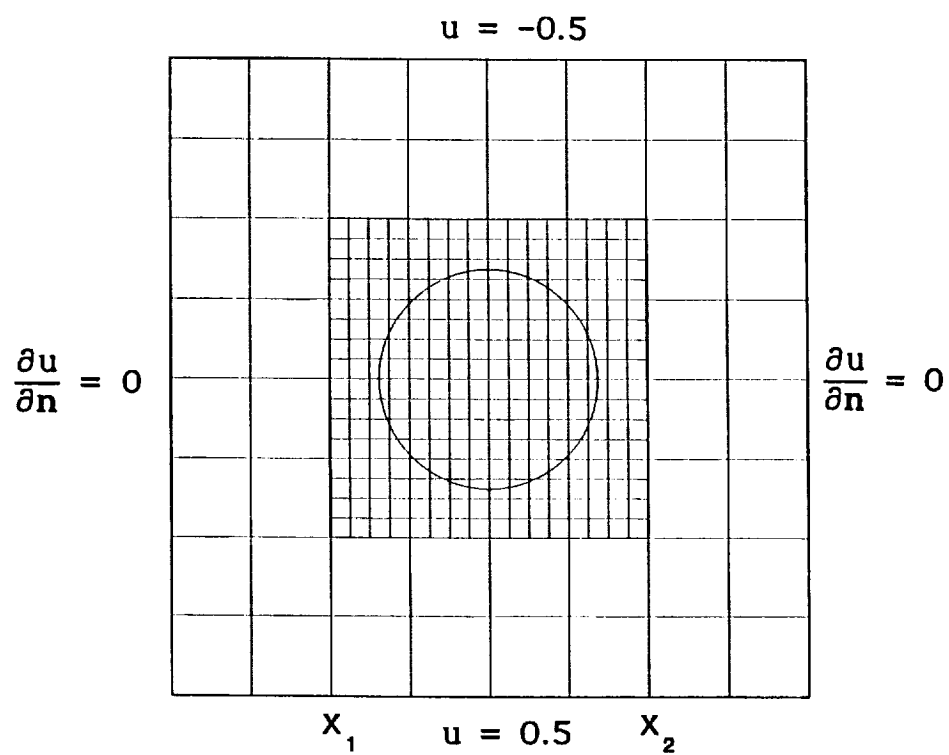


Fig. 1a. Problem (P1)

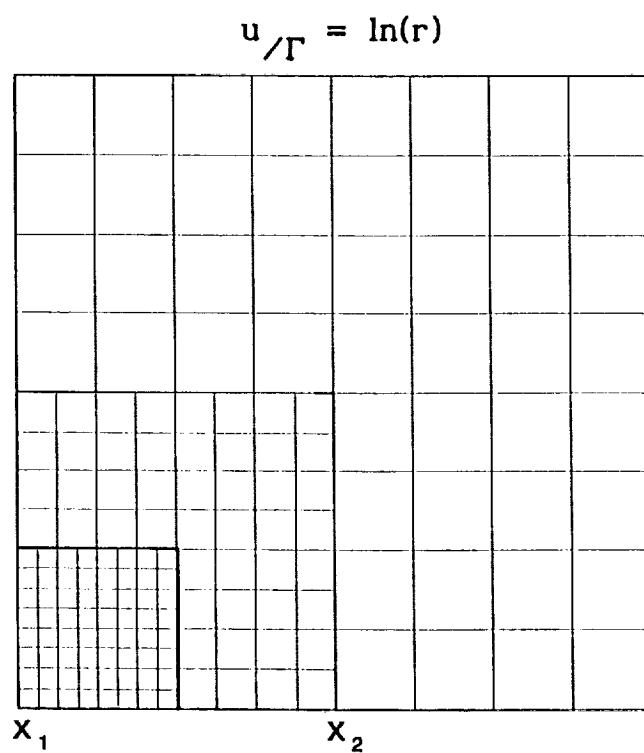


Fig. 1b. Problem (P2)

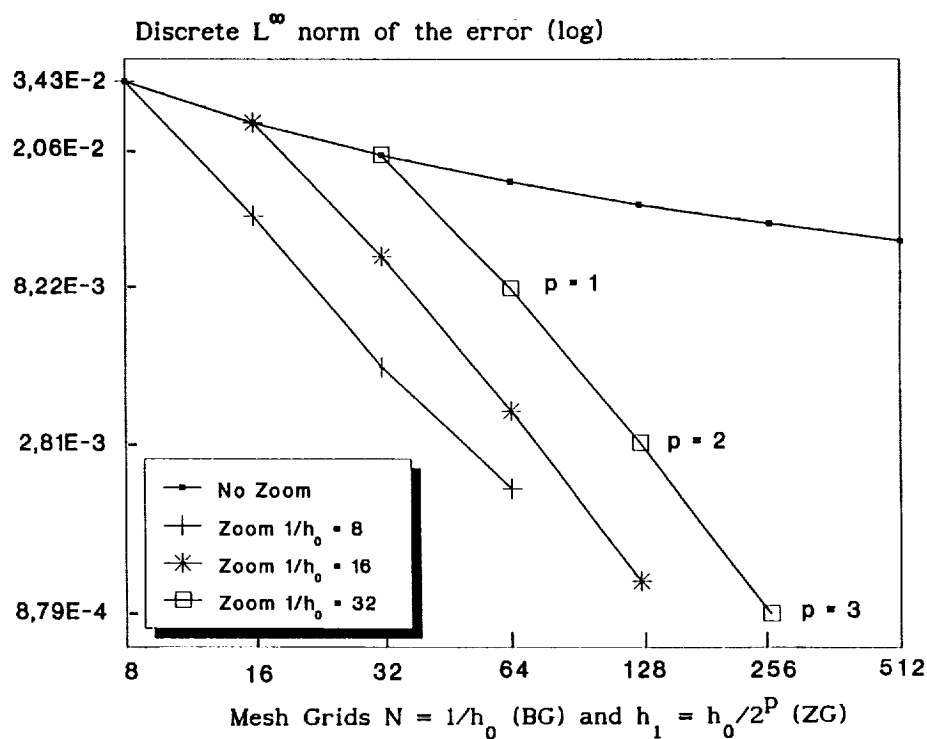


Fig. 2a. Problem (P2) - Two Grid Zoom with FIC ( $\omega = 0.35$ )  
 $\gamma(0) = 2 - x_1 = 0$  and  $x_2 = 0.5$

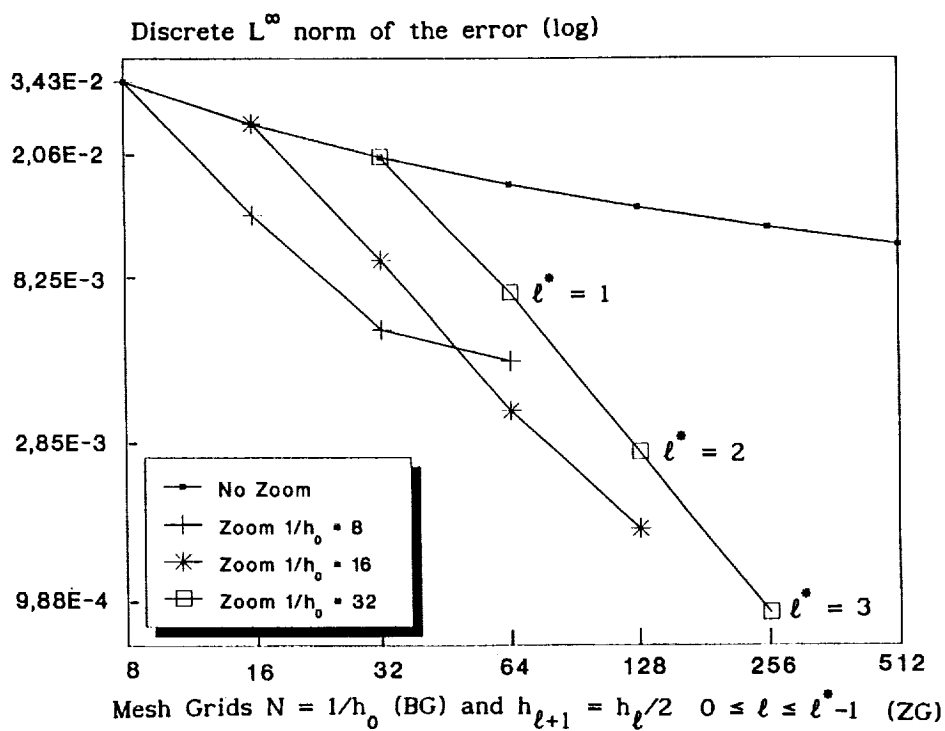


Fig. 2b. Problem (P2) - Multilevel Zoom with FIC ( $\omega = 0.35$ )  
 $\gamma(0) = 10 - x_1 = 0$  and  $x_2 = 0.5$



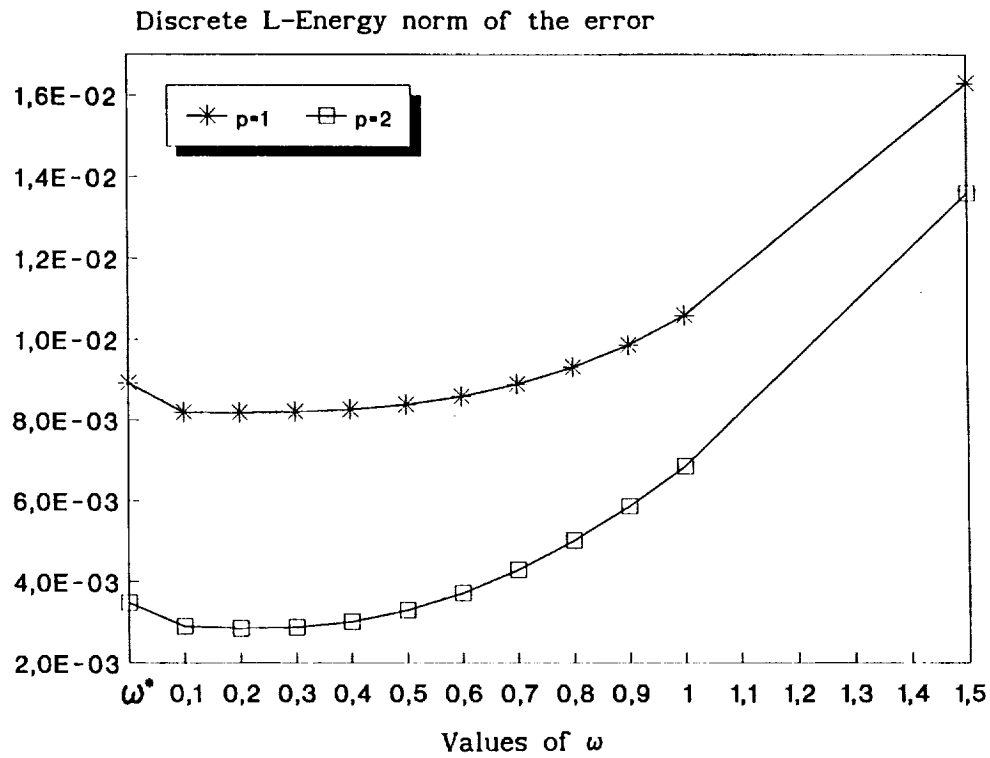


Fig. 3. Problem (P2) - Two Grid Zoom FIC - Variations with  $\omega$   
 $h_0 = 1/16$ ,  $h_1 = h_0/2^p$ ,  $\gamma(0) = 2$ ,  $x_1 = 0$  and  $x_2 = 0.5$

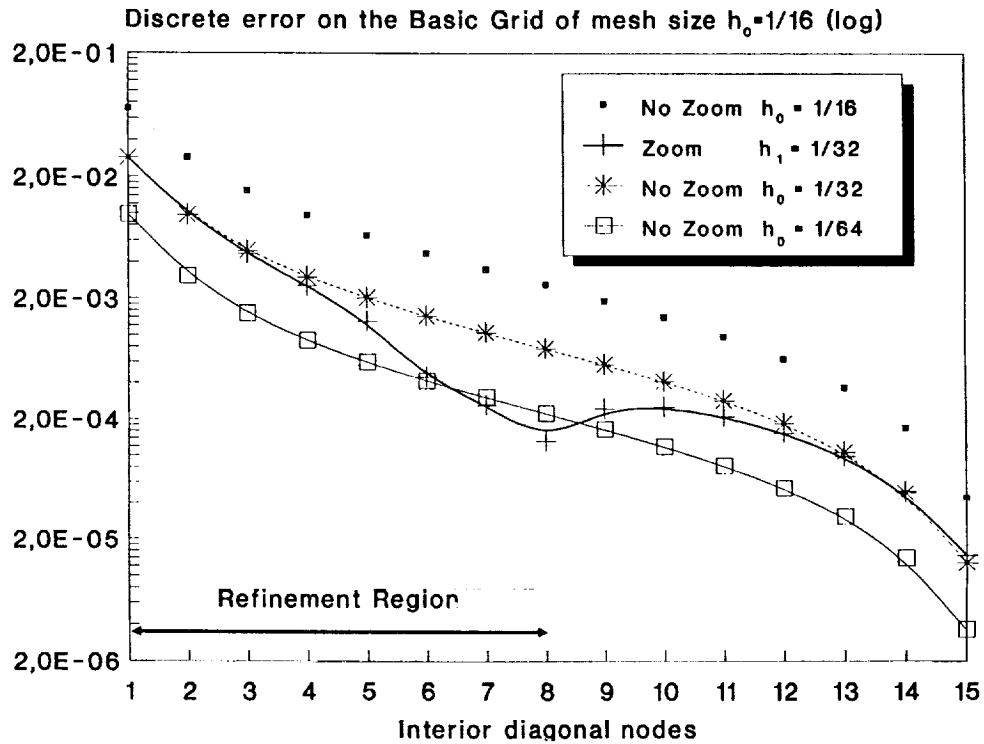


Fig. 4. Problem (P2) - Two Grid Zoom with FIC ( $\omega = 0.31$ )  
 $\gamma(0) = 2$ ,  $x_1 = 0$  and  $x_2 = 0.5$

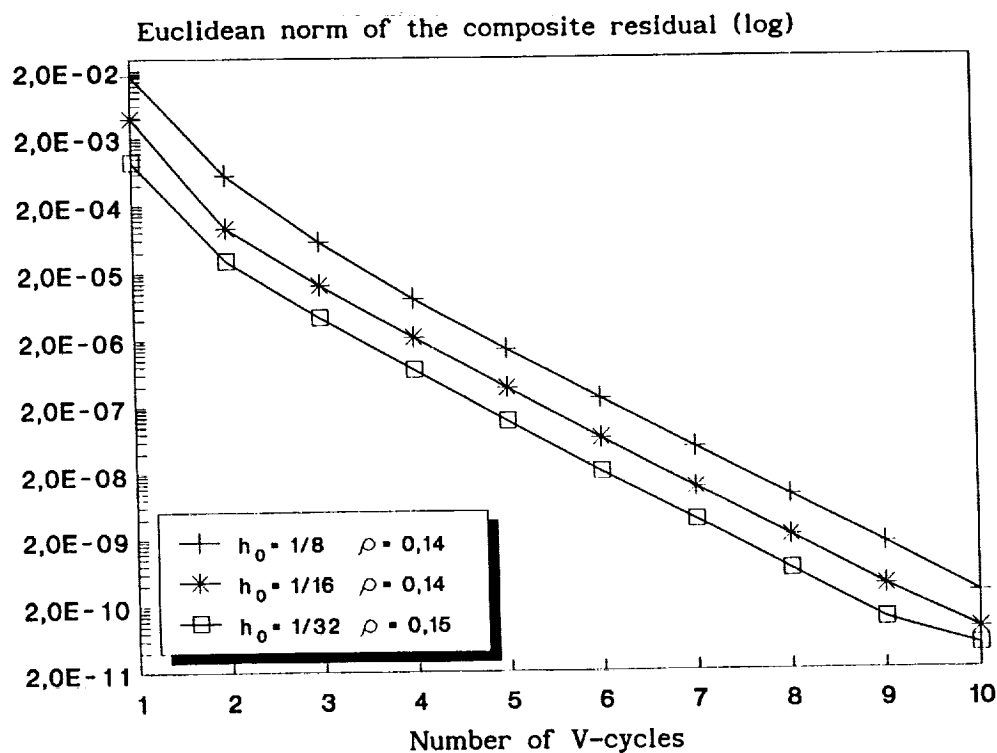


Fig. 5a. Problem (P2) - Two Grid Zoom FAC - Convergence rate  $\rho$   
 $h_1 = h_0/2$ ,  $\gamma(0) = 10$ ,  $x_1 = 0$  and  $x_2 = 0.5$

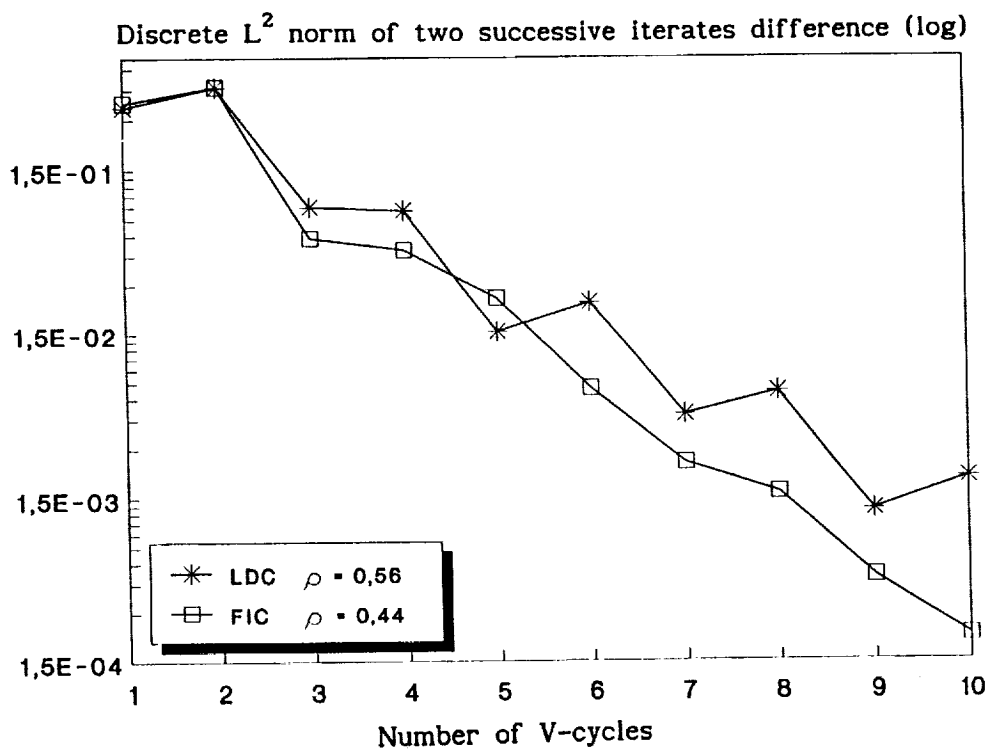


Fig. 5b. Problem (P2) - Two Grid Zoom with LDC and FIC ( $\omega = 0.2$ ) - Convergence rate  $\rho$   
 $h_0 = 1/16$ ,  $h_1 = h_0/2$ ,  $\gamma(0) = 10$ ,  $x_1 = 0$  and  $x_2 = 0.5$